The normal–ordered interaction Lagrangian for QED reads

\[ \mathcal{L}_I(x) = e : \bar{\Psi}(x) \gamma_\mu \Psi(x) A^\mu(x) :. \]

We need to decompose the fields in positive and negative frequency parts to write the time–ordered product

\[ T(\mathcal{L}_I(x)\mathcal{L}_I(y)) \]

in terms of normal–ordered products and propagators. The result should be that of Wick’s theorem.

This problem is a lengthy book–keeping task and will only be sketched here. First note that

\[ T\left( : \bar{\psi}(x_1) \gamma^\mu \psi(x_1) A_\mu(x_1) : : \bar{\psi}(x_2) \gamma^\mu \psi(x_2) A_\mu(x_2) : \right) = T\left( : \bar{\psi}_1 \gamma^\mu \psi_1 : : \bar{\psi}_2 \gamma^\nu \psi_2 : \right) T(A_{1,\mu}A_{2,\nu}) \]

since the fermion and electro–magnetic fields commute. Also, the space–time arguments have been replaced by indices, for simplicity. The second factor is simple:

\[ T(A_{1,\mu}A_{2,\nu}) = : A_{1,\mu}A_{2,\nu} : + \overline{A_{1,\mu}A_{2,\nu}} \]

To deal with the fermion part we first split off the \( \gamma \)–matrices

\[ T\left( : \bar{\psi}_1 \gamma^\mu \psi_1 : : \bar{\psi}_2 \gamma^\nu \psi_2 : \right) = (\gamma^\mu)_{ij} (\gamma^\nu)_{lm} T\left( : \bar{\psi}_{1,i} \psi_{1,j} : : \bar{\psi}_{2,l} \psi_{2,m} : \right) \]

Then we have to deal with

\[ T\left( : \bar{\psi}_{1,i} \psi_{1,j} : \bar{\psi}_{2,l} \psi_{2,m} : \right) = \theta(t_1 - t_2) : \bar{\psi}_{1,i} \psi_{1,j} : : \bar{\psi}_{2,l} \psi_{2,m} : + \theta(t_2 - t_1) : \bar{\psi}_{2,l} \psi_{2,m} : : \bar{\psi}_{1,i} \psi_{1,j} : \]

Next we decompose each spinor into positive and negative frequency components

\[ \psi_j(x) = \psi_j^{(+)}(x) + \psi_j^{(-)}(x) \quad \text{and} \quad \bar{\psi}_j(x) = \bar{\psi}_j^{(+)}(x) + \bar{\psi}_j^{(-)}(x). \]

In explicit form they are

\[ \psi_j^{(+)}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{m}{E} \sum_a b_a(k) u_j^{(a)}(k) e^{-ik \cdot x} \]

\[ \psi_j^{(-)}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{m}{E} \sum_a d_a(k) v_j^{(a)}(k) e^{ik \cdot x} \]

\[ \bar{\psi}_j^{(-)}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{m}{E} \sum_a \bar{b}_a(k) \bar{u}_j^{(a)}(k) e^{ik \cdot x} \]
\[
\psi_j(x) = \int \frac{d^3k}{(2\pi)^3} \frac{m}{E} \sum_a d_a(k) \tau^{-\alpha}_j(k) e^{-ik \cdot x}
\]

There are only two non–trivial anti–commutators \( \{ \psi_i^{(+)}(x), \psi_j^{(-)}(y) \} = U_{ij}(x - y) \) and \( \{ \psi_i^{(-)}(x), \psi_j^{(+)}(y) \} = V_{ij}(x - y) \) with

\[
U_{ij}(x - y) = \int \frac{d^3k}{(2\pi)^3} \frac{m}{E} \sum_a u_i^{(\alpha)}(k) \tau^{-\alpha}_j(k) e^{-ik \cdot (x - y)} = \int \frac{d^3k}{(2\pi)^3} \frac{e^{-ik \cdot (x - y)}}{2E} (\ell + m)_{ij}
\]

\[
V_{ij}(x - y) = \int \frac{d^3k}{(2\pi)^3} \frac{m}{E} \sum_a v_i^{(\alpha)}(k) \tau^{-\alpha}_j(k) e^{ik \cdot (x - y)} = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot (x - y)}}{2E} (\ell - m)_{ij}
\]

It is important to recall from the lecture that

\[
\theta(x^0 - y^0) \psi_{ji}(x - y) - \theta(y^0 - x^0) V_{ji}(x - y) = \langle 0 | T (\psi_j(x) \psi_i(y)) | 0 \rangle = \psi_j(x) \psi_i(y)
\]

\[
\theta(x^0 - y^0) \psi_{mi}(y - x) - \theta(y^0 - x^0) U_{mi}(y - x) = \langle 0 | T (\psi_i(x) \psi_m(y)) | 0 \rangle = \psi_i(x) \psi_m(y)
\]

This implies

\[
\overline{\psi_i(x) \psi_j(x) \psi_i(y) \psi_m(y)} = \theta(x^0 - y^0) \psi_{ji}(x - y) \psi_{mi}(y - x) - \theta(y^0 - x^0) V_{ji}(x - y) U_{mi}(y - x)
\]

With these preliminaries and

\[
: \psi_{1,i} \psi_{1,j} := (\overline{\psi_{1,i}}^{(+)} + \overline{\psi_{1,i}}^{(-)})(\psi_{1,j}^{(+)} + \psi_{1,j}^{(-)}) := \overline{\psi_{1,i}}^{(+)} \psi_{1,j}^{(+)} + \overline{\psi_{1,i}}^{(-)} \psi_{1,j}^{(+)} - \psi_{1,j}^{(-)} \overline{\psi_{1,i}}^{(+)} + \psi_{1,j}^{(-)} \overline{\psi_{1,i}}^{(-)}
\]

we write

\[
T (: \overline{\psi_{1,i}}^{(+)} \psi_{1,j}^{(+)} : : \overline{\psi_{2,l}} \psi_{2,m} : ) = \theta(t_1 - t_2) \left[ \overline{\psi_{1,i}}^{(+)} \psi_{1,j}^{(+)} + \overline{\psi_{1,i}}^{(-)} \psi_{1,j}^{(+)} - \psi_{1,j}^{(-)} \overline{\psi_{1,i}}^{(+)} + \psi_{1,j}^{(-)} \overline{\psi_{1,i}}^{(-)} \right]
\]

\[
\times \left[ \overline{\psi_{2,l}}^{(+)} \psi_{2,m}^{(+)} + \overline{\psi_{2,l}}^{(-)} \psi_{2,m}^{(+)} - \psi_{2,m}^{(-)} \overline{\psi_{2,l}}^{(+)} + \psi_{2,m}^{(-)} \overline{\psi_{2,l}}^{(-)} \right]
\]

\[
+ \theta(t_2 - t_1) \left[ \overline{\psi_{2,l}}^{(+)} \psi_{2,m}^{(+)} + \overline{\psi_{2,l}}^{(-)} \psi_{2,m}^{(+)} - \psi_{2,m}^{(-)} \overline{\psi_{2,l}}^{(+)} + \psi_{2,m}^{(-)} \overline{\psi_{2,l}}^{(-)} \right]
\]

\[
\times \left[ \overline{\psi_{1,i}}^{(+)} \psi_{1,j}^{(+)} + \overline{\psi_{1,i}}^{(-)} \psi_{1,j}^{(+)} - \psi_{1,j}^{(-)} \overline{\psi_{1,i}}^{(+)} + \psi_{1,j}^{(-)} \overline{\psi_{1,i}}^{(-)} \right]
\]

and move all negative (positive) frequency pieces to the right (left) to identify the normal–ordered form.
In the first step we assume \( t_1 > t_2 \) and start the bookkeeping:

\[
\begin{align*}
\bar{\psi}_{1,i}^+(\psi_{1,j}^+ - \psi_{1,j}^-) = & -\bar{\psi}_{1,i}^+(\psi_{1,j}^+ - \psi_{1,j}^-) = \\
& = -\bar{\psi}_{1,i}^+(\psi_{1,j}^+ - \psi_{1,j}^-) \psi_{2,t}^+ + \bar{\psi}_{1,i}^+(\psi_{1,j}^+ - \psi_{1,j}^-) \psi_{2,t}^- + \bar{\psi}_{1,i}^+(\psi_{1,j}^+ - \psi_{1,j}^-) \psi_{2,m}^+ + \bar{\psi}_{1,i}^+(\psi_{1,j}^+ - \psi_{1,j}^-) \psi_{2,m}^-.
\end{align*}
\]

Thus the terms linear in \( U \) and \( V \) are not normal-ordered.

By definition the 16 terms with neither \( U \) nor \( V \) add up to \( \bar{\psi}_{1,i}(\chi_{1,j})\psi_{1,j}^+ \bar{\psi}_{1,l}(\chi_{2})\psi_{2,m}(\chi_{2}) \), i.e. the term with no contraction in Wick's theorem. The coefficients of the term linear in \( U \) add up to

\[
\begin{align*}
: \bar{\psi}_{1,i}^+(\psi_{1,j}^+ - \psi_{1,j}^-) : + : \bar{\psi}_{1,i}^+(\psi_{1,j}^+ - \psi_{1,j}^-) : + : \bar{\psi}_{1,i}^+(\psi_{1,j}^+ - \psi_{1,j}^-) : + : \bar{\psi}_{1,i}^+(\psi_{1,j}^+ - \psi_{1,j}^-) :
\end{align*}
\]

Thus the terms linear in \( U \) contribute

\[
\begin{align*}
: \bar{\psi}_{1,i}^+(\psi_{1,j}^+ - \psi_{1,j}^-) : & = : \bar{\psi}_{1,i}^+(\chi_{1,j}) \psi_{1,j}^+(\chi_{2}) \psi_{2,m}(\chi_{2}) : \\
& = : \bar{\psi}_{1,i}^+(\psi_{1,j}^+ - \psi_{1,j}^-) :.
\end{align*}
\]

as \( t_1 > t_2 \). Similarly the terms linear in \( V \) contribute

\[
\begin{align*}
: \bar{\psi}_{1,i}^+(\psi_{1,j}^+ - \psi_{1,j}^-) : & = : \bar{\psi}_{1,i}^+(\chi_{1,j}) \psi_{1,j}^+(\chi_{2}) \psi_{2,m}(\chi_{2}) : \\
& = : \bar{\psi}_{1,i}^+(\psi_{1,j}^+ - \psi_{1,j}^-) :.
\end{align*}
\]

Hence the terms linear in \( U \) and \( V \) collect all possible two-particle contractions of fermion fields at different space–time points.

It is also obvious that the product \( U_{jl}(x_1 - x_2)V_{ml}(x_2 - x_1) \) corresponds to the contribution with all fields contracted which is displayed above since \( t_1 > t_2 \).
The second step considers $t_2 > t_1$. It is obtained from the above by appropriate substi-
tutions of the arguments and spinor indices. This and bringing back in the matrices $\gamma^\mu$
and $\gamma^\nu$ adds up to the results that

$$T (:\bar{\psi}(x_1)\gamma^\nu\psi(x_1) : : \bar{\psi}(x_2)\gamma^\mu\psi(x_2) :) = \bar{\psi}(x_1)\gamma^\mu\psi(x_1)\bar{\psi}(x_2)\gamma^\nu\psi(x_2) :$$

$$+ :\bar{\psi}(x_1)\gamma^\mu\psi(x_1)\bar{\psi}(x_2)\gamma^\nu\psi(x_2) :$$

$$+ :\bar{\psi}(x_1)\gamma^\mu\psi(x_1)\bar{\psi}(x_2)\gamma^\nu\psi_m(x_2) :$$

$$+ \bar{\psi}(x)\gamma^\mu\psi(x)\bar{\psi}(y)\gamma^\nu\psi(y),$$

which is the sum of all normal–ordered products involving all contractions of fields at
different space–time points. This includes the term without any contraction.
Finally we have to reconsider the photon field in the third step. But this just adds terms
with and without photon field contractions since the time–ordering separates between
fermion and photon fields. This verifies Wick’s theorem for QED at second order.
It is important to note that no contraction emerges between fields at coincident points
that are already normal–ordered in $\mathcal{L}_I$. 